Introduction to Derived Algebraic Geometry, I

(Trang)

<u>Classical AG</u>: here $\neq$ Italian stuff from early 1900s!

here = "before simplicial/dg comm. algs"

Let k be a commutative ring,

$CAlg_k$ = category of commutative k-algebras

not dg!

The Yoneda lemma lets us interpret a scheme as a functor:

for $X \in Sch_k$, we have

$$h_X : CAlg_k \longrightarrow Set$$
$$R \longmapsto Hom_{Sch_k}(\operatorname{Spec} R, X)$$

This functor satisfies some descent/gluing conditions (i.e., not every functor is a scheme)

Affine derived schemes (char $k=0$)

Def The ∞-category of affine derived schemes is the opposite ∞-category of $\mathrm{cdga}_k^{\leq 0}$.

Remark Given $A \in \mathrm{cdga}_k^{\leq 0}$, there is a dg ringed space $(\mathrm{Spec}\, H^0 A, \mathcal{A})$, where $\mathcal{A}$ is a sheaf of cdgas determined by A.

This gives another perspective on a derived scheme. //

We will now discuss fiber products in affine derived schemes (aka "derived intersections")

In classical case:

$$\begin{tikzcd} \mathrm{Spec}\, A_1 \times_{\mathrm{Spec}\, B} \mathrm{Spec}\, A_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & \mathrm{Spec}\, A_1 \arrow[d] \\ \mathrm{Spec}\, A_2 \arrow[r] & \mathrm{Spec}\, B \end{tikzcd} \iff \begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.O.}"] & A_1 \arrow[d] \\ A_2 \arrow[r] & A_1 \otimes_B A_2 \end{tikzcd}$$

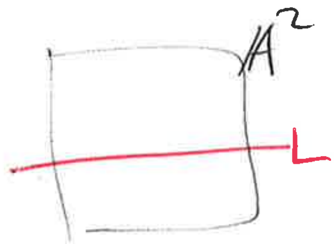
Recall that for $U, V \subseteq X$,

$$U \underset{X}{\times} V = U \cap V$$

in classical AG. So fiber product generalizes intersection.

In DAG, the derived fiber product thus encodes derived intersection.

<u>Ex</u>: $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$



$L = \{y=0\} = \operatorname{Spec}\left(k[x,y]/(y)\right)$

We want to compute $L \overset{h}{\cap} L$ in $\mathbb{A}^2$.

There is a simple resolution of A over $k[x,y]$:

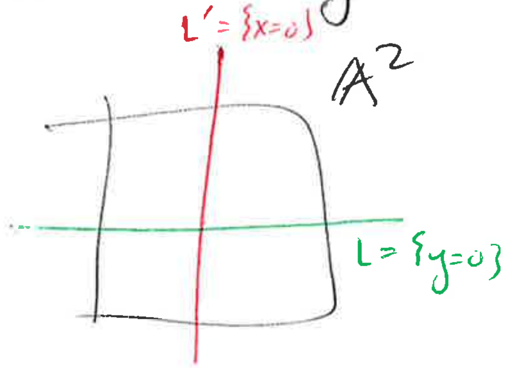
$$k[x,y] \xrightarrow{y\cdot} k[x,y] \longrightarrow k[x]$$

Let $\tilde{A} = \Big(\underbrace{k[x,y]}_{\deg 0} \underbrace{[z]}_{\deg -1}, \; d = y\frac{\partial}{\partial z} \Big)$

Then

$$\tilde{A} \underset{k[x,y]}{\otimes} A = \underset{-1}{k[x]z} \xrightarrow{\;0 \,=\, y\frac{\partial}{\partial z} \bmod y\;} \underset{0}{k[x]} = k[x,z]$$

Let's do the easy example



We can use the same resolution

$$\tilde{A} = \left(\underset{\deg -1}{k[x,y][z]},\ y\frac{\partial}{\partial z} \right)$$

Then

$$\tilde{A} \underset{k[x,y]}{\otimes} k[y] = \left(\overset{\underline{-1}}{k[y]z} \xrightarrow{y\frac{\partial}{\partial z}} \overset{\underline{0}}{k[y]} \right)$$

But this is quasi-isomorphic to k.

Upshot: the derived intersection is the classical intersection

③ if $\{U_i \to U\}_{i\in I}$ is a covering and

$\{V_{ij} \to U_i\}_{j\in J_i}$ is a covering for each U_i,

then $\{V_{ij} \to U\}_{\coprod_{i\in I} J_i}$ is a covering of U

This is a natural generalization of open coverings of a topological space (take $\mathcal{C} = \mathrm{Opens}(X)$ & usual coverings).

Ex Let $\mathcal{C} = \mathrm{Sch}_{/k}$

An étale covering of a scheme X is

$$\{\varphi_i : Y_i \to X\}$$

where $\bigcup_i \varphi_i(Y_i) = X$ and each φ_i is étale

This defines the étale topology.

Def An étale covering in $\mathrm{cdga}_k^{\leq 0}$ is a family $\{A \to B_i\}$ s.t.

① $\{\mathrm{Spec}\, H^0(B_i)\} \to \mathrm{Spec}\, H^0 A\}$ is an étale covering in the classical sense

Def Let X, Y be derived stacks.
The derived mapping stack is given by

$$\mathrm{Map}(X,Y)(\underset{\substack{\uparrow \\ \mathrm{cdga}_k^{\leq 0}}}{A}) = \mathrm{Hom}_{dSt_k}(X \times \mathrm{Spec}\, A, Y)$$

Then

Def Let $\underline{S}^1$ denote the "stackification" of the constant functor

$$\begin{aligned} \mathrm{cdga}_k^{\leq 0} &\longrightarrow \mathcal{S} \\ A &\longmapsto S^1 \end{aligned}$$

The loop space of a derived stack X is

$$\mathcal{L}X = \mathrm{Map}(\underline{S}^1, X)$$

As $S^1 \simeq * \sqcup_{S^0} *$ in $\mathcal{S}$, we see

$$\begin{aligned} \mathcal{L}X &\simeq \mathrm{Map}(* \sqcup_{S^0} *, X) \\ &\simeq \mathrm{Map}(*, X) \overset{h}{\underset{\mathrm{Map}(S^0, X)}{\times}} \mathrm{Map}(*, X) \\ &\simeq X \overset{h}{\underset{X \times X}{\times}} X \end{aligned}$$

"derived self intersection of diagonal"